Centennial Mu Alpha Theta

April 5, 2025

Algebra and Number Theory Round

Do not begin until instructed to do so.

This is the Algebra and Number Theory Round test for the 2025 DECAGON Math Tournament. You will have 50 minutes to complete 15 problems. All problems are weighted equally, but ties will be broken based on the hardest question solved (not necessarily highest numbered question). Express all answers in simplest form. Only answers recorded on the answer sheet below will be scored. Only writing tools and plain scratch paper are allowed. Assume all questions are in base 10 unless otherwise indicated. We designed this test so that most people will not be able to finish all the questions in time, so don't worry if you are struggling! Feel free to skip questions and come back to them later.

Name: ________________________ Competitor ID: ______ Team ID: ______

1. ______________ 2. ______________ 3. ______________

4. ______________ 5. ______________ 6. ______________

7. ______________ 8. ______________ 9. ______________

10. ______________ 11. ______________ 12. ______________

13. ______________ 14. ______________ 15. ______________

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1. Aryan checks the time in his afternoon class. He notices that it is a palindrome, in 12-hour time, and also a palindrome in 24-hour time. If he checked the time at $a:bc$ pm, find $100a + 10b + c$.

2. The second number and third number in a geometric sequence of common ratio 3 are the same as the third number and fifth number of an arithmetic sequence respectively. If the first number of the arithmetic sequence is 5, what is the first number of the geometric sequence?

3. Victor has a large collection of tennis balls from a variety of brands. The ratio of Penn to Wilson balls is $5:3$, the ratio of Dunlop to Head balls is $7:2$, the ratio of Head to Babolat balls is $4:5$, and the ratio of Penn to Babolat balls is $8:1$. The ratio of Wilson to Dunlop balls in Victor's collection can be expressed as $a:b$. where a and b are relatively prime positive integers. Find $a+b$.

4. Homer and Bart both have some integer number of donuts and can only take integer numbers of donuts from each other. Bart takes 20% of Homer's donuts. After that, Homer then takes 75% of Bart's donuts. After this, Homer has the same number of donuts he started with. What is the least number of total donuts Bart and Homer can have?

5. Two different factors of 12 are randomly chosen. What is the probability that they are relatively prime?

6. What is the smallest 5-digit palindrome that is divisible by 11 and 3?

7. How many zeros does the number $5^{100000} \times 128!$ end with?

8. A Martian clock looks like an Earth clock with 12 evenly spaced numbers and a minute and hour hand. It starts with the hour hand pointed at 1 and the minute hand pointed at 12. Every Earth minute, if the hour hand is pointing at n, then the minute hand advances by n numbers. Once the minute hand returns back to 12, the hour hand advances by 1 number. For example, it takes 18 Earth minutes for the hour hand to point at the number 3. How many Earth minutes does it take for the clock to return to its starting position?

9. Aryan the CS Major wrote code that requires a positive integer n input. It outputs $\frac{n}{2}$ if n is even and $3n+1$ if n is odd. He runs the code 7 times, the first time inputting integer n and repeatedly inputting the previous output. The final output is 1. Find the sum of all possible n.

10. Avy has the memory of a goldfish and can only remember numbers that alternate between ***exactly*** two digits, such as 32323 or 919, but not 333. How many 15-digit numbers that are divisible by 15 can he remember?

11. Given positive integers a and b, the largest power of 2 that divides $a+b$ is 256. If the largest power of 2 that divides ab is 2^n, how many values of n less than 100 are possible?

12. In the coordinate plane, a circle centered at the point $(0, 1)$ has a radius of 1 unit. What is the largest value of a such that $y = ax^2$ intersects the circle exactly once?

13. Let S denote the set of positive integers of the form $2^a \cdot 3^b$ for positive integers a and b. Compute

$$\sum_{n \in S} \frac{1}{n^2}$$

(Here $\sum_{n \in S} \frac{1}{n^2}$ means "the sum of $\frac{1}{n^2}$ for all n in the set S".)

14. Suppose $r_1, r_2, \ldots, r_{2025}$ are the roots of the polynomial

$$x^{2025} - 2025x - 1$$

Find the value of

$$\sum_{i=1}^{2025} \sum_{j=i+1}^{2025} \frac{r_i}{r_j} + \frac{r_j}{r_i}$$

15. Hexadecimal (base-16) numbers are written with digits 0 to 9 and A to F to represent 10 to 15. Hexatrigesimal (base-36) numbers are written with digits 0 to 9 and A to Z to represent 10 to 35. Let n be the number of positive integers less than 1000 that can be represented in hexadecimal with only digits 0 to 9 and m be the number of positive integers less than 1000 that can be represented in hexatrigesimal with only digits 0 to 9. Find $n + m$.

Algebra and Number Theory Answers

1. 131
2. $-\frac{5}{3}$
3. 19
4. 16
5. $\frac{7}{15}$
6. 13431
7. 127
8. 77
9. 190
10. 2
11. 91
12. $\frac{1}{2}$
13. $\frac{1}{24}$
14. -2025
15. 498

Algebra and Number Theory Solutions

1. The answer is $\boxed{131}$. Since it is the afternoon, in 24-hour time, the time is of the form $1a : bc$. Yet this must be a palindrome, so $c = 1$ and $a = b$. Thus, the time is of the form $1a : a1$. In 12-hour time, the time then must be of the form $d : a1$. Yet this is a palindrome as well, so $d = 1$; in particular, it is between 1 and 2 o'clock.

   So $1a : a1$ is between 1 and 2 o'clock, forcing $a = 3$. So the time is $1 : 31$ pm.

2. Let a be the first item of the geometric sequence, and d be the common difference of the arithmetic sequence. We get $5 + 4d = 9a$ and $5 + 2d = 3a$. Solving gives a $= \boxed{-\dfrac{5}{3}}$.

3. Let p be the number of Penn balls, w be the number of Wilson balls, d be the number of Dunlop balls, h be the number of Head balls, and b be the number of Babolat balls. We are given that $\frac{p}{w} = \frac{5}{3}$, $\frac{d}{h} = \frac{7}{2}$, $\frac{h}{b} = \frac{4}{5}$, and $\frac{p}{b} = \frac{8}{1}$. To find $\frac{w}{d}$, we first observe that we want a w in the numerator, which we can get by taking the reciprocal of the first ratio: $\frac{w}{p} = \frac{3}{5}$. Next, we want to change the p in the denominator to a d. Since we must cancel several variables in succession, the following sequence of operations will get to our answer: $\frac{w}{p} \times \frac{p}{b} \times \frac{b}{h} \times \frac{h}{d} = \frac{w}{d}$. Substituting gives us $\frac{3}{5} \times \frac{8}{1} \times \frac{5}{4} \times \frac{2}{7} = \frac{12}{7}$. The sum of the numerator and denominator is thus $12 + 7 = \boxed{19}$.

4. Let h be the number of donuts Homer starts with, and let b be the number of donuts Bart starts with. Homer ends with $\frac{4}{5}h + \frac{3}{4}(b + \frac{h}{5}) = h$. We get the equation $h = 15b$. The smallest integers are $h = 15$ and $b = 1$, yielding an answer of $\boxed{16}$.

5. 12 is equal to $2^2 * 3^1$, so it has $(2 + 1)(1 + 1) = 6$ factors, and $6 * 5 = 30$ ordered pairs of different factors. We can then consider each prime factor separately. Starting with 2, if we only choose factors from 2^2, the two numbers can be $(1, 1)$, $(2, 1)$, $(4, 1)$, $(1, 2)$, or $(1, 4)$ for 5 combinations. Similarly for 3, they can be $(1, 1)$, $(1, 3)$, or $(3, 1)$ for 3 combinations. We can multiply them together to get 15 combinations because the prime factors of two numbers independently influence the GCF. However, we must subtract 1 because we cannot include $(1, 1)$, giving a final probability of $\frac{15-1}{30} = \boxed{\dfrac{7}{15}}$.

6. Let's express our 5-digit palindrome as *abcba* where a, b, and c, are digits. Since our number cannot start with 0, the smallest possible value for a is 1. To be divisible by 3, $2a + 2b + c \equiv 0 \pmod 3$ or $2b + c \equiv 1 \pmod 3$. To be divisible by 11, $2a + c = 2b$ or $c + 2 = 2b$. We can test out every value of b and c, starting with the minimum possible $c = 0$, and we find that $b = 3$ and $c = 4$ are the smallest values that satisfy both. Thus, our answer is $\boxed{13431}$.

7. The number of zeros that the number ends with is the number of tens you can factor out of the number. The number of tens you can factor out of the number is the number of twos and fives you can pair up with each other. Clearly we have more than enough fives. The number of twos that you can factor out out of 128! is

   $$64 + 32 + 16 + 8 + 4 + 2 + 1 = 127.$$

   64 for each number divisible by 2, 32 for each divisible by 4, 16 for each divisible by 8, etc. This is fine since 4 has 2 powers of 2 so we're counting it twice and 8 has 3 powers of 2 and we're counting it 3 times. So the answer is $\boxed{127}$.

8. Notice that it will take $\frac{12}{\gcd(n,12)}$ Earth minutes for the hour hand to move advance one number if it started at n. Thus, the total time it takes is

   $$\sum_{n=1}^{12} \frac{12}{\gcd(n, 12)} = \sum_{n=1}^{12} \frac{4 * 3}{\gcd(n, 4)\gcd(n, 3)} = \sum_{n=1}^{4} \frac{4}{\gcd(n, 4)} * \sum_{n=1}^{3} \frac{3}{\gcd(n, 3)} = (4 + 2 + 4 + 1)(3 + 3 + 1)$$

   which equals $\boxed{77}$. Alternatively you can just bash and add all 12 numbers.

9. Working backwards, we can start with the final output and list all possible numbers in each intermediate step. So, this would look like: $\{1\}$(last output) $\leftarrow \{2\} \leftarrow \{4\} \leftarrow \{1,8\} \leftarrow \{2,16\} \leftarrow \{4,5,32\} \leftarrow \{1,8,10,64\} \leftarrow \{2,16,3,20,21,128\}$(first input). The sum is $\boxed{190}$.

10. The 15 digit numbers Avy can remember must be in the form $5A5A5A5A5A5A5A5$ to with $5+A+5+A+5+A+5+A+5+A+5+A+5+A+5 \equiv 0 \mod 3$ to be divisible by 15 (divisible by 5 and 3). Simplifying the equation, we get $40+7A \equiv 0 \mod 3 \to 7A \equiv 2 \mod 3 \to A \equiv 2 \mod 3$. Since A is a digit less than 10, the only digits A can be are $2, 5, 8$. We cannot have 5 because we are already using 5 for the other repeating digit, so there are $\boxed{2}$ options left.

11. Let $v_2(x)$ be the exponent of the largest power of 2 that divides x. Then $n = v_2(ab) = v_2(a) + v_2(b)$. Notice that if a is divisible by 256, then it is required that $v_2(b) \geq 9$ since $v_2(a+b) = 8$. Then, every $n \geq 17$ satisfies the criteria, giving 83 values. If $0 \leq v_2(a) < 8$, then $v_2(a) = v_2(b)$ so every even number n in $[0, 14]$ works, giving 8 values. Thus, there are a total of $83 + 8 = \boxed{91}$ values of n.

12. The equation of the circle is $x^2 + (y-1)^2 = 1$ and we want it to intersect $y = ax^2$ exactly once. The x^2 in the first equation can be replaced with $\frac{y}{a}$ to get $\frac{y}{a} + (y-1)^2 = 1$ and then $\frac{y}{a} + y^2 - 2y = 0$. The only way for this equation to have exactly one root is to be equivalent to $y^2 = 0$. It follows that $\frac{y}{a} - 2y = 0$ for all y and therefore $a = \boxed{\frac{1}{2}}$

13. The answer is $\frac{1}{24}$. Instead of using the set S, consider how n is just any product of a power of 2 and a power of 3. Thus, it is equivalent to summing $\frac{1}{(2^a \cdot 3^b)^2}$ across all positive integers a and positive integers b. It follows that

$$\sum_{n\in S} \frac{1}{n^2} = \sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{1}{(2^a \cdot 3^b)^2} = \sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{1}{4^a \cdot 9^b}$$

Consider fixing a and considering $\sum_{b=1}^{\infty} \frac{1}{4^a \cdot 9^b}$. Since a is fixed, then $\frac{1}{4^a}$ is just a constant, so

$$\sum_{b=1}^{\infty} \frac{1}{4^a \cdot 9^b} = \frac{1}{4^a}\sum_{b=1}^{\infty} \frac{1}{9^b} = \frac{1}{4^a} \cdot \frac{1}{8}$$

where we used the geometric series formula that

$$\sum_{b=1}^{\infty} \frac{1}{9^b} = \frac{1}{9} + \frac{1}{9^2} + \cdots = \frac{\frac{1}{9}}{1 - \frac{1}{9}}$$

It follows that

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{1}{4^a \cdot 9^b} = \sum_{a=1}^{\infty} \frac{1}{4^a} \cdot \frac{1}{8} = \frac{1}{8}\sum_{a=1}^{\infty} \frac{1}{4^a} = \frac{1}{8} \cdot \frac{1}{3} = \frac{1}{24}$$

where we used the geometric series formula again. Thus, the answer is $\boxed{\frac{1}{24}}$.

14. The answer is -2025. The requested sum looks something like

$$\begin{aligned}
&\frac{r_1}{r_2} + \frac{r_1}{r_3} + \cdots + \frac{r_1}{r_{2025}} \\
&+\frac{r_2}{r_1} + \frac{r_2}{r_3} + \cdots + \frac{r_2}{r_{2025}} \\
&+\ldots \\
&+\frac{r_{2025}}{r_1} + \frac{r_{2025}}{r_2} + \cdots + \frac{r_{2025}}{r_{2024}}
\end{aligned}$$

let this sum be S. With some wishful thinking, we may try to factor this as

$$(r_1 + r_2 + \cdots + r_{2025}) \left(\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_{2025}} \right)$$
$$= S + \frac{r_1}{r_1} + \frac{r_2}{r_2} + \cdots + \frac{r_{2025}}{r_{2025}}$$

this is true, because every fraction $\frac{r_i}{r_j}$ for any $1 \leq i, j \leq 2025$ (possibly $i = j$) appears when the left-hand side is expanded; it is the result of the r_i in the first factor multiplying with r_j in the second factor.

But by Vieta's, $r_1 + r_2 + \cdots + r_{2025} = 0$, as it is the negation of the coefficient of x^{2024}. It follows that

$$0 = S + \frac{r_1}{r_1} + \frac{r_2}{r_2} + \cdots + \frac{r_{2025}}{r_{2025}} \implies S = \boxed{-2025}$$

as claimed.

15. The maximum number in hexadecimal that satisfies these conditions is 399_{16}, and the maximum number in hexatrigesimal that satisfies these conditions is 99_{36}. Excluding 0, the total number of digits that can be represented is $399 + 99 = \boxed{498}$.