Centennial Mu Alpha Theta

April 5, 2025

Combinatorics Round

Do not begin until instructed to do so.

This is the Combinatorics Round test for the 2025 DECAGON Math Tournament. You will have 50 minutes to complete 15 problems. All problems are weighted equally, but ties will be broken based on the hardest question solved (not necessarily highest numbered question). Express all answers in simplest form. Only answers recorded on the answer sheet below will be scored. Only writing tools and plain scratch paper are allowed. Assume all questions are in base 10 unless otherwise indicated. We designed this test so that most people will not be able to finish all the questions in time, so don't worry if you are struggling! Feel free to skip questions and come back to them later.

Name: ______________________ Competitor ID: ______ Team ID: ______

1. ____________ 2. ____________ 3. ____________

4. ____________ 5. ____________ 6. ____________

7. ____________ 8. ____________ 9. ____________

10. ____________ 11. ____________ 12. ____________

13. ____________ 14. ____________ 15. ____________

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1. An ant starts at the origin of a coordinate plane and wants to crawl to the point $(3, 3)$. If it can only take steps right or up, how many paths can the ant take?

2. What is the probability that when 8 people standing in a circle each flip a fair coin, there exists two people standing adjacent to each other such that both flip heads or both flip tails?

3. In the United Provinces of Skyrim, there are 10 provinces numbered 1 to 10, with province n having a population of n^2 people. Each province gets n tokens. If a majority of the population in a province votes for you, you get all of the tokens of the province. What is the minimum amount of votes to get more than half of the total tokens?

4. Howser is attacking Marco. When he attacks, he independently and randomly chooses one out of 3 tactics to use, throwing hammers, spitting fire, or jumping on Marco. Throwing hammers has a $1/4$ chance of hitting Marco, spitting fire has a $2/3$ chance of hitting and jumping on him has a $1/12$ chance of hitting. What is the probability that after 5 attacks, Howser has hit Marco at least once?

5. In a game, whenever a hero is hit, they randomly and independently add either 1, 2, or 3 to their block meter. If their block meter reaches or exceeds 8, then they block. What is the probability that a hero blocks after exactly 7 hits?

6. Kevin is fixing a bookshelf with 10 books. The books are in a line and have pairwise distinct heights. Whenever Kevin finds two books a, b such that a is to the left of b yet a is taller than b, he swaps the two. He stops when he can no longer swap two books. Let m denote the minimum number of moves he needs and M denote the maximum number of moves he needs. Find $m + M$.

7. Kevin is given a sequence of 5 distinct numbers from left to right selected from 1 to 9 (inclusive) but then quickly forgets them. However, he remembers that the average of the numbers is included in the list, and that the *complement* of the sequence is the same as the original sequence when read from right to left. (The *complement* of a sequence is formed by replacing each number x with $10 - x$. For example, the complement of $1, 4, 5, 6, 9$ is $9, 6, 5, 4, 1$.)

   Find the number of possible sequences of numbers that Kevin could have originally had.

8. Kevin is going to start working out! He plans to spend three days in the gym, but he is inconsistent; each day (excluding the first), he flips a fair coin; if it is heads, he spends double the amount of time he spent yesterday; if it is tails, he spends 15 minutes less than the time he spent yesterday. On the first day, he trains for 30 minutes. What is the expected number of minutes he trains on the third day, to the nearest integer number of minutes?

9. In the game Mateo Carts, you can hold up to 2 items. When you open an item box while you're not holding any items, there's a $\frac{2}{3}$ chance to get junk and a $\frac{1}{3}$ chance to get a powerup. However, if you're already holding junk, you are guaranteed to get a powerup in your next box. When you get a powerup, you automatically use it and discard all items you're holding. You start without any items. What is the expected number of powerups you get after opening 3 boxes?

10. Chris, Zak, and Vinay each take three tests, where each test is out of 30 points and a score can range from 0 to 30 (inclusive). We say a person A is *barely better* than another person B if for at least one of the tests, A scored better than B.

    Suppose Chris is barely better than Zak, and Zak is barely better than Vinay. A person's score is the sum of their scores on the three tests. If Chris's score is m and Vinay's score is M, what is the maximum value of $M - m$?

11. Alfredo and Remy are sharing a circular pizza. Alfredo cuts the pizza into 10 unequal slices, ranking them by size from 1 (smallest) to 10 (largest). They take turns eating, with Remy going first. Remy may choose any slice for his first move. After that, each player must eat a slice adjacent to an already eaten slice. Remy always selects the largest available slice on his turn. Alfredo, knowing this, cuts the pizza to maximize the sum of the rankings of the slices he eats. What is the sum of the rankings of the slices Alfredo eats?

12. In the AMC 10 competition, there are 25 questions with the following scoring:

    - 6 points for a correct answer
    - 0 points for an incorrect answer
    - 1.5 points for an unanswered question

    for a maximum of 150 points. How many scores are achievable with exactly one combination of correct, incorrect, and unanswered? For example, 99 and 127.5 are achievable with multiple combinations, while 142.5 is not achievable at all.

13. The letters in the word "DECAGON" are rearranged so that no two vowels are adjacent. How many strings of letters can be formed, including the original "DECAGON"?

14. In a 4×4 grid of points, how many lines can be drawn that pass through two or more points?

15. An ant starts in a corner cell of a 4x4 grid of unit squares. At any step, it can move to any cell a distance of 1 unit away (but not diagonally). In how many ways can the ant visit all the cells and return to its original square in exactly 16 steps?

Combinatorics Answers

1. 20
2. $\frac{127}{128}$
3. 75
4. $\frac{211}{243}$
5. $\frac{20}{2187}$
6. 45
7. 48
8. 49
9. 43/27
10. 88
11. 31
12. 16
13. 1440
14. 62
15. 12

Combinatorics Solutions

1. To reach $(3,3)$, the ant must make 3 right and 3 up steps, while the order does not matter. This means there are a total of $\binom{6}{3} = \boxed{20}$ paths.

2. The answer is $\frac{127}{128}$. We use complementary counting.

   Assume that no two people who are adjacent get the same result. Suppose that that people are labeled $1, 2, \ldots, 8$, where i is adjacent to $i+1$ for $i = 1, 2, \ldots, 7$ and 8 is adjacent to 1. Assume that the results of the flips are H or T, and represent the resulting flips as $A_1A_2A_3A_4A_5A_6A_7A_8$, where A_i is the result for i. Then

   $$A_1A_2A_3A_4A_5A_6A_7A_8 = HTHTHTHT \text{ or } A_1A_2A_3A_4A_5A_6A_7A_8 = THTHTHTH$$

   there are 2^8 total possible results, with 2 possibilities for each person. So the probability in the complementary case is $\frac{2}{2^8} = \frac{1}{128}$.

   It follows that the desired probability is $1 - \frac{1}{128} = \boxed{\dfrac{127}{128}}$.

3. In general, the smallest provinces are the most efficient for population per token. There are $\frac{10\times 11}{2} =$ 55 total tokens, so you need at least 28 tokens to have a majority. Winning the first 7 provinces gives you 28 tokens. Winning the first 7 provinces need $1 + 3 + 5 + 9 + 13 + 19 + 25 = \boxed{75}$ votes.

4. When Howser attacks, he has a $\frac{1}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{1}{12} = \frac{1}{3}$ chance of hitting Marco. The chance he hits Marco at least once is 1 - (the chance he hits Marco 0 times). He has a $\frac{2}{3}$ chance to not hit Marco on any given attack, so the chance he misses all 5 attacks is $(\frac{2}{3})^5$ which is $\frac{32}{243}$. 1 - $\frac{32}{243} = \frac{211}{243}$

5. Blocking in 7 hits needs very low numbers, you either get 6 1s in a row, then get a 2 or 3 at the end, or get 5 1s and a 2 and then any number at the end. The chance of getting 6 1s in a row then a 2 or 3 is $(\frac{1}{3})^6 \times \frac{2}{3} = \frac{2}{2187}$. The chance of getting 5 1s and a 2 is $\frac{6}{729}$ (The 2 can be any of the first 6 numbers) $\frac{2}{2187} + \frac{18}{2187} = \frac{20}{2187}$.

6. We can model this as sorting the books from left to right and in ascending height. When the books are already sorted there is nothing to do, so $m = 0$.

   1 3 2 4 5 6 7 8 9 10

   The key idea for the upper bound is to consider swapping adjacent books. Intuitively, this means it takes more moves; rigorously, if we count the number of **inversions** (the number of pairs of books such that the left one is taller than the right one), which effectively measures "disorder", the total number of inversions must decrease and by only swapping adjacent books it decreases by just 1 each time. An example of an inversion is highlighted in red above.

   10 9 8 7 6 5 4 3 2 1

   Thus the upper bound is just the maximum number of inversions in the original order; intuitively, it is when the books are in reverse order and we can only swap two books at a time. Then it takes 9 moves to move 1 to the front, 8 moves to move 2, etc. and the answer is $0 + 1 + \cdots + 9 = \boxed{45}$.

7. The answer is 48. Let the sequence be a, b, c, d, e. Then we have that this must be the same as $10-e, 10-d, 10-c, 10-b, 10-a$, thus $a+e = b+d = 10$ and $c = 5$. Furthermore, then the average is

   $$\frac{a+b+c+d+e}{5} = \frac{10+10+5}{5} = 5$$

   and indeed, $c = 5$, so the average is in the list.

   Consider the pairs $(1,9), (2,8), (3,7), (4,6)$. Then (a,e) and (b,d) must be one of these pairs, possibly reversed. There are $4 \cdot 3$ ways to choose the pairs, and then $2 \cdot 2$ ways to order them. Thus the total number of possibilities is $12 \cdot 4 = \boxed{48}$.

8. The answer is 49. We present two ways to approach the problem.

Solution via casework. There is a $\frac{1}{2}$ chance that Kevin trains for 60 minutes on day 2. Then Kevin expects to train $\frac{1}{2}(2 \cdot 60) + \frac{1}{2}(60 - 15) = 82.5$ minutes on day 3.

There is a $\frac{1}{2}$ chance that Kevin instead trains for 15 minutes on day 2. Then Kevin expects to train for $\frac{1}{2}(2 \cdot 15) + (15 - 0) = 15$ minutes on day 3.

Considering both cases, Kevin expects to spend

$$\frac{1}{2} \cdot 82.5 + \frac{1}{2} \cdot 15 = 48.75$$

minutes in the gym on day 3. The closest integer is 49.

Solution via linearity of expectation. Let X_2, and X_3 be random variables representing the number of minutes he spends in the gym on day 2 and day 3, respectively. Suppose $\mathbb{E}[X]$ denotes the expected value of the random variable X. Then using the fact that day 2 depends on day 1,

$$\mathbb{E}[X_2] = \frac{1}{2}(2 \cdot 30) + \frac{1}{2}(30 - 15) = 37.5$$

Now, notice that from similar reasoning, we also have

$$\mathbb{E}[X_3] = \frac{1}{2}(\mathbb{E}[2 \cdot X_2]) + \frac{1}{2}(\mathbb{E}[X_2 - 15])$$

due to Linearity of Expectation, we have that

$$\mathbb{E}[X_3] = \frac{1}{2}(2 \cdot \mathbb{E}[X_2]) + \frac{1}{2}(\mathbb{E}[X_2] - 15) = \frac{3}{2}\mathbb{E}[X_2] - \frac{15}{2}$$

It follows that

$$\mathbb{E}[X_3] = \frac{3}{2} \cdot 37.5 - \frac{15}{2} = 48.75$$

the closest integer is 49.

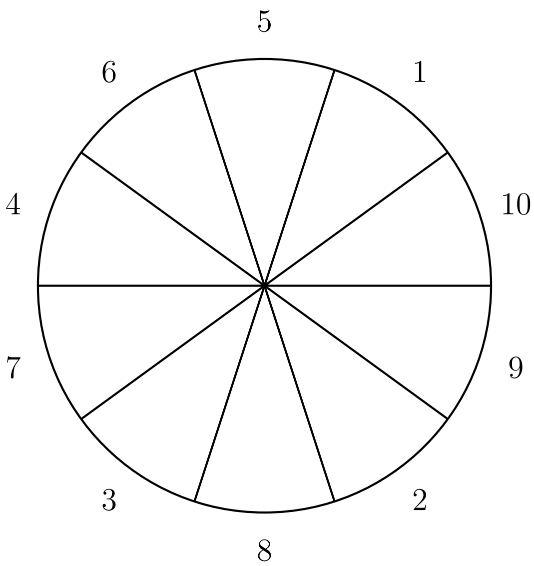
9. Let J represent getting junk, let P represent getting a powerup. After opening 3 boxes you can get PPP, JPP, PPJ, PJP, or JPJ. There's a $\frac{1}{3}^3$ to get PPP, $\frac{2}{3} \times \frac{1}{3}$ to get JPP. $\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3}$ to get PPJ, $\frac{1}{3} \times \frac{2}{3}$ to get PJP, and $\frac{2}{3} \times \frac{2}{3}$ to get JPJ. The expected value is the number of powerups you get in a specific case multiplied by the chance for that case to happen which is: $\frac{3}{27} + \frac{4}{9} + \frac{4}{27} + \frac{4}{9} + \frac{4}{9} = \frac{43}{27}$.

10. The answer is 88. Let (a, b, c) represent scores of a, b and c on the three tests respectively.

    Note that Chris cannot obtain a score of $(0, 0, 0)$; then it is impossible for Chris to do barely better than Zak. Similarly, Vinay cannot obtain a score of $(30, 30, 30)$, because then it is impossible for Zak to be barely better.

    Thus, $m \geq 1$ and $M \leq 89$, hence $M - m \leq 89 - 1 = 88$. This can be obtained when Chris scores $(1, 0, 0)$, Zak scores $(0, 30, 30)$, and Vinay scores $(30, 30, 29)$. Thus, $\boxed{88}$ is indeed the maximum.

11. Clearly, Remy will eat the largest slice (slice 10) first, so Alfredo must position the slices based on this information. By putting slice 9 adjacent to slice 10, Alfredo can eat it, and then putting 1 and 2 around that section will force Remy to eat slice 2. Alfredo can similarly force Remy to eat slices 3 and 4 while eating slices 5 and 6 himself. In the end, he eats slices 1,9,8,7 and 6 which add to $\boxed{31}$. Optimal arrangement:



Note: This is provably the maximum since Remy will always eat slice 10 and Alfredo will always be forced to eat slice 1 and the largest slices left are 9,8,7,6.

12. We do casework based on the number of problems attempted (not unanswered). If all 25 problems are attempted, we can get 0, 1 or 2 problems wrong to get a score that satisfies the condition, for a total of 3 scores. Getting a score with 3 problems wrong can also be achieved by leaving 4 problems unanswered, so most lower scores also don't satisfy the condition. This argument can also be applied to 24, 23, or 22 problems attempted for a total of $4 * 3 = 12$ scores. All four cases yield different scores because they all leave different remainders when divided by 6 (0, 1.5, 3, and 4.5).
However, the scores on the lower end: 0, 1.5, 3, and 4.5 also satisfy the condition, since they can only achieved with 0 problems correct. Thus, there are a total of $\boxed{16}$ scores that can only be achieved with one combination.

13. Replace vowels with 'A' and consonants with 'B' to get 'BABABAB'. If we start with the string 'AAA', we insert consonants between the vowels to satisfy the condition: 'ABABA'. Then we can insert the remaining 2 consonants however we want to make a new string. Define a group as the space separated by vowels so that there are 4 groups, since there are 3 vowels. There are 4 ways to put the consonants if they are in the same group and $\binom{4}{2} = 6$ ways if they are not. For each of these arrangements, there are 3! ways to order the vowels and 4! for the consonants, for a total of $3! * 4! * (4 + 6) = \boxed{1440}$ strings.

14. There are 16 total vertices so there are $\binom{16}{2}$ line segments that can be formed. However, we need to subtract the cases where 3 or more vertices line on a line. There are 4 lines that contain exactly 3 vertices (45° diagonal to the grid) and 10 lines containing exactly 4 vertices (4 horizontal, 4 vertical, 2 diagonal). The lines containing exactly 3 vertices contain $\binom{3}{2}$ line segments, but we must subtract all but 1 of them to correctly count it. The same thing can be applied to the lines with 4 vertices. Thus, you can form a total of $\binom{16}{2} - 4(\binom{3}{2} - 1) - 10(\binom{4}{2} - 1) = \boxed{62}$ lines.

15. To solve this problem, we can find the number of looping paths that passes through every cell exactly once, then multiply by 2 because there are two directions the ant can start moving in. First, notice that corner cells in the path must be connected to the two adjacent cells. Then there are just the four center cells left to traverse. They can either be 4 adjacent cells or 2 groups of 2 adjacent cells in the path. The first option has 4 ways and the second has 2, for a total of $2(4 + 2) = \boxed{12}$.