Centennial Mu Alpha Theta

April 5, 2025

Estimathon Round

Do not begin until instructed to do so.

This is the Estimathon Round test for the 2025 DECAGON Math Tournament. You will have 30 minutes to complete 8 problems. All problems are weighted equally, but ties will be broken based on the hardest question solved (not necessarily highest numbered question). Express all answers in simplest form. Only answers recorded on the answer sheet below will be scored. Only writing tools and plain scratch paper are allowed. Assume all questions are in base 10 unless otherwise indicated. We designed this test so that most people will not be able to finish all the questions in time, so don't worry if you are struggling! Feel free to skip questions and come back to them later.

All answers must be real numbers. Answers that are not real numbers receive zero points.

Names: ________________________ Competitor ID: ______ Team ID: ______

1. _____________ 2. _____________ 3. _____________

4. _____________ 5. _____________ 6. _____________

7. _____________ 8. _____________

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1. Estimate the value of

$$\sqrt{1+98\sqrt{1+99\sqrt{1+100\sqrt{1+\ldots}}}}$$

If your answer is E and the correct answer is A, you will receive $\left(\min\left(\frac{E}{A}, \frac{A}{E}\right)\right)^4$ points.

2. Let the set $S = \{1, 3, 6, 10 \ldots\}$ be the infinite set of all triangular numbers. Estimate

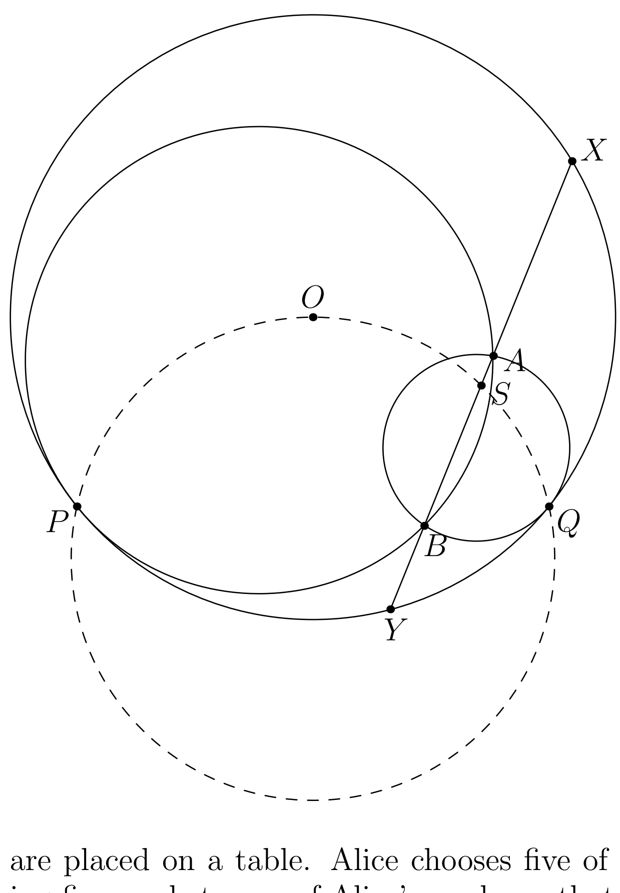
$$\sum_{n \in S} \frac{1}{n^3}$$

where the summation is taken over all elements in S.

If your answer is E and the correct answer is A, you will receive $e^{-|E-A|}$ points.

3. A circle Ω with center O and radius 5 is internally tangent to two circles γ_1 and γ_2 at P and Q respectively, such that γ_1 and γ_2 are both inside Ω. Suppose γ_1 and γ_2 intersect at points A and B, and that AB intersects Ω again at points X and Y. It is given that $XY = 7$. Let the circumcircle of $\triangle OPQ$ intersect XY again at S. Estimate OS.

If your answer is E and the correct answer is A, you will receive $\max\left(0, \frac{\sin(\ln\frac{E}{A})}{\ln(\frac{E}{A})}\right)$ points.



4. 10 cards numbered 1 to 10 are placed on a table. Alice chooses five of these cards. Bob then tries to match each of the remaining five cards to one of Alice's cards, so that the matched card is greater than the card Alice chose. For example, if Alice chose cards $1, 2, 4, 7, 9$, then Bob could match the cards $3, 5, 6, 8, 10$ to $1, 2, 4, 7, 9$ respectively.

It turns out that Bob cannot do this. Estimate the number of ways Alice could have chosen her cards.

If your answer is E and the correct answer is A, you will receive $\left(\min\left(\frac{A}{E}, \frac{E}{A}\right)\right)^2$ points.

5. There are positive reals a, b that satisfy the systems of equations

$$a + \frac{a}{a^2+b^2} = \frac{2}{\sqrt{3}}$$

$$b - \frac{b}{a^2+b^2} = \frac{4\sqrt{2}}{\sqrt{7}}$$

Estimate $a+b$.

If your answer is E and the correct answer is A, you will receive $\max\left(0, 1 - \left(\frac{E}{A} - 1\right)^2\right)$ points.

6. For some prime p with $p < 1000$, there exists exactly two integers a with $0 < a < p$ such that p divides a^2+1. Estimate the number of possible p.

If your answer is E and the correct answer is A, you will receive $\max\left(0, 1 - \left(\frac{|E-A|}{A}\right)^4\right)$ points.

7. An ant starts at the center of a circle with radius 1. Every second, it moves 1 unit a random direction. What is the average time, in seconds, that it will take for the ant to leave the circle? (the ant does not leave the circle after the first second)

If your answer is E and the correct answer is A, you will receive $\left(\min\left(\frac{E}{A}, \frac{A}{E}\right)\right)^4$ points.

8. A function $f(x)$ takes in an gives out an integer. If x is even, it outputs $\frac{x}{2}$. If x is odd, it outputs $3x+1$. Let $f_n(x) = f(f_{n-1}(x))$, where $f_1(x) = f(x)$. The Collatz conjecture states that, for any positive integer x, $f_n(x) = 1$ for some finite n. Estimate the smallest value of n such that $f_n(2025) = 1$.

If your answer is E and the correct answer is A, you will receive $\max\left(0, 1 - \left(\frac{E}{A} - 1\right)^2\right)$ points.

Estimathon Answers

1. 99
2. $80 - 8\pi^2$
3. $\frac{\sqrt{51}}{2}$
4. 210
5. $\frac{\sqrt{7}+2}{\sqrt{21}} + \frac{\sqrt{8}+\sqrt{14}}{\sqrt{7}}$
6. 80
7. 2.5598
8. 156

1. By radical expansion,

$$n = \sqrt{1 + (n-1)\sqrt{1 + (n)\sqrt{1 + (n+1)\sqrt{1 + ...}}}}$$

thus the answer is $n = \boxed{99}$.

2. Every element in S can be written in the form $n = \frac{x(x+1)}{2}$ for $x \geq 1$. Then, $\frac{1}{n} = \frac{2}{x(x+1)}$, which can be rewritten as $2(\frac{1}{x} - \frac{1}{x+1})$. So $\frac{1}{n^3} = 8(\frac{1}{x} - \frac{1}{x+1})^3 = 8(\frac{1}{x^3} - \frac{1}{(x+1)^3} - \frac{3}{x(x+1)^2} + \frac{3}{x(x+1)^2})$.

   The last two terms of the expression can be combined into a single fraction $\frac{3x-3(x+1)}{x^2(x+1)^2} = -\frac{3}{(x(x+1))^2}$. This looks exactly like the original expression of n,
   turning it into $-3(\frac{1}{x} - \frac{1}{x+1})^2 = -3(\frac{1}{x^2} + \frac{1}{(x+1)^2} - \frac{2}{x(x+1)})$. But again, that last term is basically just n, so rewrite it one last time as $-3(\frac{1}{x^2} + \frac{1}{(x+1)^2} - \frac{2}{x} + \frac{2}{x+1})$

   Finally, simplifying heavily, we obtain the equation

$$8(\sum_{x=1}^{\infty}(\frac{1}{x^3} - \frac{1}{(x+1)^3} - 3\sum_{x=1}^{\infty}(\frac{1}{x^2} + \frac{1}{(x+1)^2}) + 6\sum_{x=1}^{\infty}(\frac{1}{x} - \frac{1}{x+1})) \tag{1}$$

   In the first summation, every term except the first cancels out with another term so that the whole thing simplifies to 1. The same applies to the third. The second summation can be rewritten as

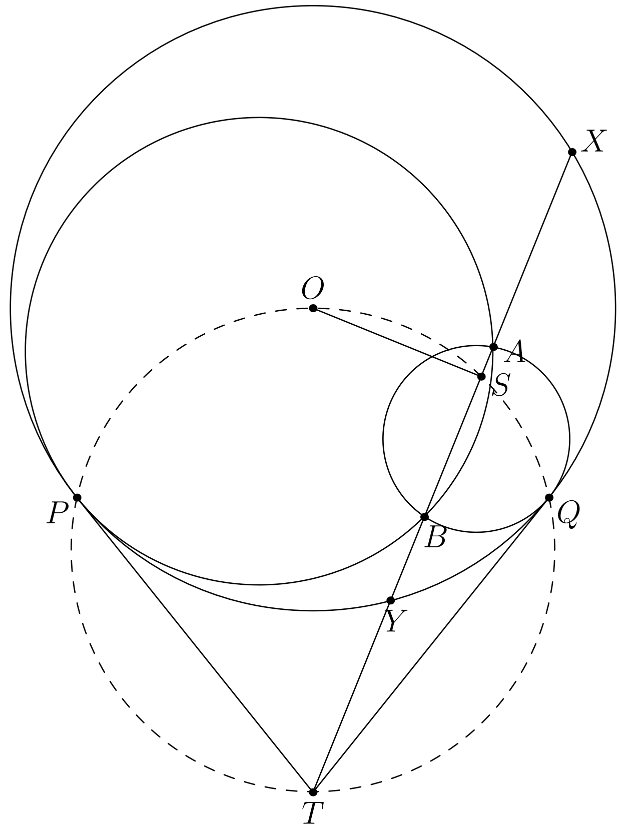
$$\sum_{x=1}^{\infty}\frac{1}{x^2} + \sum_{x=1}^{\infty}\frac{1}{(x+1)^2} = 2\sum_{x=1}^{\infty}\frac{1}{x^2} - 1 = \frac{\pi^2}{3} - 1 \tag{2}$$

   because the sum of the inverse of all squares is $\frac{\pi^2}{6}$.

   Finally, the entire expression is simplified to

$$8(1 - 3(\frac{\pi^2}{3} - 1) + 6(1)) = \boxed{80 - 8\pi^2}$$

3. Consider the following diagram:



Note by Radical Axis theorem that the tangent to Ω at P, the tangent to Ω at Q, and XY concur. Let this point of concurrence be T. Notice that then since TP and TQ are the tangents from T to Ω, $\angle OPT = \angle OQT = 90$ and so $OPTQ$ is cyclic with diameter OT.

It follows that $\angle OST = 90$ so S is the foot from O to XY. Then S is the midpoint of XY since XY is a chord. Then $OX = 5$ and $SX = \frac{7}{2}$, so by Pythagorean theorem, $OS = \boxed{\frac{\sqrt{51}}{2}}$.

4. The answer is $\boxed{210}$.

First, notice that if we assume that Alice's cards are in sorted order $a_1 < a_2 < a_3 < a_4 < a_5$, then it is always advantageous for Bob to match the lowest card with a_1, the second lowest with a_2, etc.; assume for the sake of contradiction that Bob matches X to a_i and Y to a_j, where $i < j$ yet $X > Y$. Then $a_i < a_j < Y < X$ and so then we could just swap the two.

Thus, we may instead consider the process as follows:

- Starting with the first card, assign to it the letter A or B, symbolizing if it goes to Alice's hand or Bob's hand;
- If a card is assigned B, then try to match it to the lowest card labeled A that has not been matched

The key claim is that at, if at some point the cards $1, 2, \ldots, n$ have been assigned letters, then the number of cards assigned A must be greater than or equal to the number of cards assigned B. Assume for the sake of contradiction there are more cards assigned B; then one of them cannot be matched with a card labeled A, contradiction.

Thus, we can rephrase the problem yet again:

- Start at $(0, 0)$ on the coordinate plane, and make ten moves, where each time we either go up or to the right, and we end on $(5, 5)$;
- Go to the right on move i if card i is assigned A, and move up on move i if card i is assigned B;

- The path drawn must always stay on or below the line $y = x$.

The first bullet point is equivalent to 5 cards being dealt to both players. The second and third bullet point correspond to our previous claim.

But the number of such paths is well-known to be the 5th Catalan number, where the nth Catalan number is given by $\frac{1}{n+1}\binom{2n}{n}$. A sketch of a proof is to show the recurrence

$$C_n = C_{n-1} + C_1 C_{n-2} + C_2 C_{n-3} + \cdots + C_{n-2} C_1 + C_{n-1}$$

where C_n is the nth Catalan number and there are n summands.

It follows that the answer is $\binom{10}{5} - \frac{1}{6} \cdot \binom{10}{5} = 210$.

5. The answer is $\boxed{\frac{\sqrt{7}+2}{\sqrt{21}} + \frac{\sqrt{8}+\sqrt{14}}{\sqrt{7}}}$. The idea is that if we add the first equation to i times the second equation, where $i = \sqrt{-1}$, then

$$a + bi + \frac{a - bi}{a^2 + b^2} = \frac{2}{\sqrt{3}} + \frac{4\sqrt{2}}{\sqrt{7}} i$$

and now if $z = a + bi$ where z is a complex number, then $\frac{a-bi}{a^2+b^2} = \frac{\overline{z}}{|z|^2} = \frac{1}{z}$. Thus, we have

$$z + \frac{1}{z} = \frac{2}{\sqrt{3}} + \frac{4\sqrt{2}}{\sqrt{7}} i$$

which rearranges to

$$z^2 - \left(\frac{2}{\sqrt{3}} + \frac{4\sqrt{2}}{\sqrt{7}} i\right) z + 1 = 0$$

this is a quadratic in z; we find that

$$z = \frac{\frac{2}{\sqrt{3}} + \frac{4\sqrt{2}}{\sqrt{7}} i \pm \sqrt{\left(\frac{2}{\sqrt{3}} + \frac{4\sqrt{2}}{\sqrt{7}} i\right)^2 - 4}}{2}$$

expanding,

$$\left(\frac{2}{\sqrt{3}} + \frac{4\sqrt{2}}{\sqrt{7}} i\right)^2 - 4 = \frac{4}{3} - \frac{32}{7} + \frac{16\sqrt{2}}{\sqrt{21}} i - 4$$

$$= -\frac{152}{21} + \frac{16\sqrt{2}}{\sqrt{21}} i$$

suppose that this quantity is equal to $(\sqrt{r} + \sqrt{s} i)^2$. So we have

$$-\frac{152}{21} + \frac{16\sqrt{2}}{\sqrt{21}} i = (\sqrt{r} + \sqrt{s} i)^2 = r^2 - s^2 + 2rsi$$

thus we have $r^2 - s^2 = -\frac{152}{21}$ and $rs = \frac{8\sqrt{2}}{\sqrt{21}}$. We may solve to find that $r = \frac{4}{\sqrt{21}}$ and $s = 2\sqrt{2}$, thus

$$\left(\frac{4}{\sqrt{21}} + 2\sqrt{2} i\right)^2 = \left(\frac{2}{\sqrt{3}} + \frac{4\sqrt{2}}{\sqrt{7}} i\right)^2 - 4$$

it follows that

$$z = \frac{\frac{2}{\sqrt{3}} + \frac{4\sqrt{2}}{\sqrt{7}} i \pm \left(\frac{4}{\sqrt{21}} + 2\sqrt{2} i\right)}{2} = \frac{1}{\sqrt{3}} + \frac{2\sqrt{2}}{\sqrt{7}} i \pm \left(\frac{2}{\sqrt{21}} + \sqrt{2} i\right)$$

due to the answer form, it follows that

$$z = a + bi = \left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{21}}\right) + \left(\frac{2\sqrt{2}}{\sqrt{7}} + \sqrt{2}\right) i$$

we find that $a = \frac{\sqrt{7}+2}{\sqrt{21}}$ and $b = \frac{\sqrt{8}+\sqrt{14}}{\sqrt{7}}$.

6. The answer is $\boxed{80}$. We first consider the existence of an integer $0 < a < p$ with $p \mid a^2 + 1$. Thus, we have

$$a^2 \equiv -1 \pmod{p}$$

and so this implies

$$a^4 \equiv 1 \pmod{p}$$

observe that by Fermat's Little Theorem,

$$a^{p-1} \equiv 1 \pmod{p}$$

Thus, it follows that

$$a^{\gcd(4,p-1)} \equiv 1 \pmod{p}$$

Note that $\gcd(4, p-1) = 1, 2$ or 4. Yet if $\gcd(4, p-1) = 1$ then we have $a = 1$ and then $p \mid 2$, which contradicts the fact that there are two such integers. If $\gcd(4, p-1) = 2$ then $a^2 \equiv 1 \pmod{p}$, so $p \mid a^2 - 1$. Yet $p \mid a^2 + 1$ so $p \mid 2$ here, contradiction again.

Thus $\gcd(4, p-1) = 4$. In other words, $4 \mid p-1$. Now, I claim that for any prime p with $4 \mid p-1$, then there are exactly two values of a with $p \mid a^2 + 1$. Indeed, both a and $p - a$ work, and

$$a^2 \equiv b^2 \pmod{p} \implies a \equiv b, -b \pmod{p}$$

thus we cannot have $a^2 \equiv b^2 \pmod{p}$ for $a < b < \frac{p}{2}$.

The number of primes p less than 1000 with $4 \mid p - 1$ is 80.

7. The answer is $\boxed{2.5598}$. We use the following code:

```
import numpy as np

def simulate_exit_steps(num_simulations):
    steps_to_exit = []

    for _ in range(num_simulations):
        x, y = 0.0, 0.0
        steps = 0

        while np.sqrt(x**2 + y**2) <= 1.0:
            angle = np.random.uniform(0, 2 * np.pi)
            x += np.cos(angle)
            y += np.sin(angle)
            steps += 1

        steps_to_exit.append(steps)

    return steps_to_exit


def calculate_confidence_interval(data, confidence=0.95):
```

```
    mean = np.mean(data)
    std_err = np.std(data, ddof=1) / np.sqrt(len(data))
    z_score = 1.96
    margin_of_error = z_score * std_err
    return mean, (mean - margin_of_error, mean + margin_of_error)


num_simulations = 10_000_000  # Number of simulations to run

steps_to_exit = simulate_exit_steps(num_simulations)

mean_steps, confidence_interval = calculate_confidence_interval(steps_to_exit)

print(f"Average number of steps to exit: {mean_steps:.4f}")
```

8. The answer is $\boxed{156}$. We use the following code:

```
def collatz(n):
    steps = 0
    cur = n
    while cur != 1:
        if cur % 2 == 0:
            cur = cur // 2
        else:
            cur = 3 * cur + 1
        steps += 1
    return steps

print(f"collatz for 2025: {collatz(2025)}")
```