Centennial Mu Alpha Theta

April 5, 2025

General Round

Do not begin until instructed to do so.

This is the General Round test for the 2025 DECAGON Math Tournament. You will have 50 minutes to complete 15 problems. All problems are weighted equally, but ties will be broken based on the hardest question solved (not necessarily highest numbered question). Express all answers in simplest form. Only answers recorded on the answer sheet below will be scored. Only writing tools and plain scratch paper are allowed. Assume all questions are in base 10 unless otherwise indicated. We designed this test so that most people will not be able to finish all the questions in time, so don't worry if you are struggling! Feel free to skip questions and come back to them later.

Name: ________________________ Competitor ID: ______ Team ID: ______

1. ______________ 2. ______________ 3. ______________

4. ______________ 5. ______________ 6. ______________

7. ______________ 8. ______________ 9. ______________

10. ______________ 11. ______________ 12. ______________

13. ______________ 14. ______________ 15. ______________

Centennial Mu Alpha Theta

April 5, 2025

General Round

1. Find the area of the triangle with vertices at the coordinates $(3, 1)$, $(4, -1)$, and $(-2, 5)$.
2. How many 4-digit numbers can be formed using the digits $1, 2, 3, 4, 5$ without repetition such that the number is greater than 3000?
3. An integer n can be written as a 3-digit integer in base 4. If n is divisible by 3, how many possible values of n are there?
4. There are 9 six-digit numbers whose digits are all equal (such as 111111 and 222222). Find the largest prime number that is a divisor of all of them.
5. Bob has a calculator with only two buttons: One button adds 1 while the other button multiplies by 3. Assuming he starts at the number 0, what is the minimum number of button presses for him to reach the number 47?
6. How many ways can 4 different 6-sided dice be rolled such that the sum of all 4 numbers is odd?
7. $\log_2 3 \times \log_3 4 \times \log_4 5 \times ... \times \log_{2024} 2025$ can be written as $a \log_2 b$, where a and b are positive integers and b is as small as possible. Find $a + b$.
8. An ant and a crumb of food are each located at a random position on the circumference of a unit circle. In order to get to the food, the ant must travel along the circumference. What is the average distance the ant must travel to reach the food?
9. In a square with side lengths 2, a line segment is drawn between two opposite corners. Three circles of equal radius are placed with their centers on this line. The first circle is tangent to exactly two sides of the square, the second circle is tangent to the other two sides of the square, and the last circle is at the center of the square and is externally tangent to the other two circles. Find the radius of the circles.
10. What is the remainder when $10^{10} + 10^{16} + 10^{22} + ... + 10^{64}$ is divided by 7?
11. Find the number of five-digit numbers that contain all distinct digits that satisfy the following:
    - The number formed is divisible by 5
    - The number contains at least one even digit
    - The first digit cannot be 0
12. A 4 by 4 chess board, essentially a 4 by 4 grid of squares, has two indistinguishable white bishops and one black rook occupying distinct squares. A bishop is a piece that can move any number of squares diagonally, and a rook is a piece that can move any number of squares horizontally and vertically. We say that a piece can be captured if another piece of a different color can move to the square that the piece is occupying. How many ways can we arrange the two white bishops and one black rook such that no piece can be captured by another piece in one move?
13. Albert and Caleb are running around a circular track in opposite directions. Since Albert is much taller than Caleb and therefore faster, he gives Caleb a five-second head start. Despite this, Albert is still able to run $\frac{5}{8}$ of the circumference of the track before he first bumps into Caleb. If it takes an average of 2 minutes for Caleb to complete a full lap around the track, how many seconds would it take Albert to complete a full lap?

14. How many ways can the letters of the word "STATISTICS" be arranged such that no two S's are adjacent?

15. Ten cards placed on a table have the numbers 1 through 10 written on them. In each turn, Nathaniel randomly removes a card from the table and stops if the number on the card is prime. What is the expected number of cards Nathaniel removes?

General Round Answers

1. 3
2. 72
3. 16
4. 37
5. 8
6. 648
7. 47
8. $\frac{\pi}{2}$
9. $\sqrt{2}-1$
10. 5
11. 5688
12. 204
13. 64
14. 23520
15. $\frac{11}{5}$

General Round Solutions

1. To find the area of any polygon defined by the coordinates of its vertices, we can use the Shoelace Theorem. The formula in this case would be $A = \frac{1}{2}|(x_1y_2+x_2y_3+x_3y_1)-(y_1x_2+y_2x_3+y_3x_1)|$. Plugging in our coordinates gives us $A = \frac{1}{2}|(3\times-1+4\times5+-2\times1)-(1\times4+-1\times-2+5\times3)| = \frac{1}{2}|15-21| = \boxed{3}$.

2. We can systematically calculate the possible numbers for each digit. There are only 3 choices for the first digit, as the number has to be greater than 3000. There are 4 possible numbers for the second digit, 3 for the third and 2 for the fourth. This leads us to a total number of $3 \cdot 24 = \boxed{72}$ solutions.

3. We can express n as a 3-digit base 4 number abc_4, where a, b, and c are digits between 0 and 3, inclusive. The smallest and largest values that satisfy this are 100_4 and 333_4, which equal 16 and 63 in base 10, respectively. So, $16 \leq n \leq 63$. There are $\boxed{16}$ multiples of 3 in this range.

4. Since 111111 is the greatest common divisor in all these values, it would be wise to consider the prime factorization of 111111. As $111111 = 111 * 1001 = 3 * 7 * 11 * 13 * 37$, the answer is $\boxed{37}$

5. If we write the number 47 in base 3, we get 1202. Let us think backward. Instead of adding 1 and multiplying by 3 starting from 0, we subtract 1 and divide by 3 starting from 47. Let "-" denote pressing the subtraction by 1 button and "/" denote pressing the division by 3 button. Note that if we ever have —/ in our button presses, we can replace it with a /- to make it shorter. Additionally, the only way we can divide by 3 is if the unit digit is 0. We cannot press the subtraction button more than 2 times in a row if the unit digit is not 0 as then we can make it shorter, so we must divide by 3 as soon as we see a 0 in the unit place. This results in the button presses "–//–/-", totaling $\boxed{8}$ button presses.

6. We know that adding an even number of odd numbers or all even numbers will result in even, so the only way to get an odd number is to roll an odd number of even numbers. In the case that there is only 1 odd die, we would choose 1 die out of the 4 to be odd, which can be done in $\binom{4}{1} = 4$ ways. Each odd die has 3 possible values (1, 3, or 5), and each of the remaining 3 even dice also has 3 possible values (2, 4, or 6). To find the total number of possible outcomes for this case, we multiply: $4 \times 3 \times 3^3 = 324$. For the case of 3 odd dice, we would choose 3 of the 4 dice to be odd, which has $\binom{4}{3} = 4$ different ways. The remaining die has 3 possible values, allowing for $4 \times 3^3 \times 3 = 324$. Adding the two gets us $\boxed{648}$.

7. Using the properties of logarithms, specifically change of base,

$$\log_a b \times \log_b c = \frac{\log b}{\log a} \times \frac{\log c}{\log b} = \frac{\log c}{\log a} = \log_a c$$

then applying the above repeatedly, we can simplify the expression into $\log_2 2025$, which can be simplified into $2\log_2 45$, so the answer is $\boxed{47}$.

8. Imagine two points on the unit circle where one is θ radians clockwise of the other, where $0 \leq \theta < 2\pi$. Then the circumferential distance between the two points is θ if $0 \leq \theta < \pi$ and $2\pi - \theta$ otherwise. Therefore the distance the ant travels is between 0 and π, half the circumference of the circle. Since the locations are picked at random, the expected value of this piecewise function is $\boxed{\frac{\pi}{2}}$

9. If we split the square in half along the line that passes through the diameters of all 3 circles, we get a $45-45-90$ triangle with 3 semicircles along the hypotenuse. We can represent the length of the hypotenuse in terms of the radius r. We can form a $45-45-90$ triangles with a radii of the outer circles and the sides of the square, and the hypotenuse of that triangle is $\sqrt{2} \times r$. Therefore, the hypotenuse is equal to $4 \times r + 2 \times (\sqrt{2} \times r)$. This gives us equation $(4+2\sqrt{2})r = 2\sqrt{2}$, and after solving and simplifying, we get $\boxed{\sqrt{2}-1}$

10. We consider the powers of 10 and their remainders divided by seven:

$$\begin{aligned} 10 &= 7+3 \\ 10^2 &= 7\cdot 10 + 3\cdot 10 = 7\cdot 14 + 2 \\ 10^3 &= 7\cdot 140 + 20 = 7\cdot 142 + 6 \\ &\dots \end{aligned}$$

Continuing the pattern, notice that the remainder of 10^k is the remainder of $10\cdot r$, where r is the last remainder. Then the remainder of 10^{10} is 4, and the pattern goes $3, 2, 6, 4, 5, 1$.

Then, notice that every term contributes a remainder of $4 \mod 7$, and there are 10 terms total, the total remainder would be the remainder of $40 \equiv 5 \pmod 7$, thus the answer is $\boxed{5}$.

11. In order to ensure that the number is divisible by 5, we must check two cases: if the last digit is 0 or 5. This gives us $9\cdot 8\cdot 7\cdot 6 = 3024$ numbers with the last digit of 0 and $8\cdot 8\cdot 7\cdot 6 = 2688$ numbers if the last digit is 5. However, we also need to consider the constraint that at least one digit is even. We do not need to check the numbers ending in 0, as they already contain an even digit, but we must subtract all numbers without an even digit if the last digit is 5. This includes all permutations of the first 4 digits of 13795, which there are 24. This gives a new total of $3024 + 2688 - 24 = \boxed{5688}$.

12. We first place the rook. Note there are 3 different places to put it in due to symmetry: the corners, the edges but not the corners, and the center. Note that wherever we place the rook, the squares covered diagonally, horizontally, and vertically by the rook cannot be placed with a bishop. The corner cases give us $4\times\binom{6}{2}$, the edges but not the corner cases give us $8\times\binom{6}{2}$, and the center cases give us $4\times\binom{4}{2}$, totaling $\boxed{204}$.

13. We can express this problem in terms of distance-rate-time equations $d = rt$. Let us define d as the full circumference of the track, r_a as Albert's rate, r_c as Caleb's rate, and t as the time between when Caleb started running and when they bumped into each other. Since Albert ran $\frac{5}{8}$ the length of the track and started 5 seconds late, his equation is $\frac{5}{8}d = r_a(t-5)$. For Caleb to meet in the same location, he must've ran $\frac{3}{8}$ of the track, so his equation is $\frac{3}{8}d = r_c t$. Lastly, we know that Caleb can run the whole track in 2 minutes, or 120 seconds, so this can be represented as $d = 120r_c$. Plugging this into Caleb's first equation and cancelling r_c gives us $t = 45$, and plugging that into Albert's equation gives us $\frac{5}{8}d = 40r_a$ then $d = 64r_a$. Thus, Albert can complete a lap in $\boxed{64}$ seconds.

14. To ensure that no two S's are adjacent, we will be using the Gap Method. First, we will consider the number of ways to arrange the Non-S Letters. Since there are 7 non-S's, 3 T's, 2 I's, 1 A, and 1 C, the number of arrangements would be $7!/(3!\cdot 2!) = 420$. Now we have to identify the number of arrangements to place the S's. As there are 8 gaps and 3 S's, $\binom{8}{3} = 56$. The total valid arrangements would be: $420 * 56 = \boxed{23520}$

15. There are 4 cards whose number is prime, so we can label those as prime and the other cards as not prime, since the exact value of the prime doesn't matter. The problem can also be interpreted as shuffling the cards in a random order counting what position the first prime numbered card appears. If the first prime is in position 7, there are $\binom{3}{3}$ ways to order the 3 other primes in the 3 positions after the first. This continues down the line until position 1, where there are $\binom{9}{3}$ ways. There are $\binom{10}{4}$ total ways to arrange 10 cards, so the expected value is:

$$\frac{7\binom{3}{3} + 6\binom{4}{3} + \dots\binom{9}{3}}{\binom{10}{4}} = \frac{\binom{4}{4} + \binom{5}{4} + \dots\binom{10}{4}}{\binom{10}{4}} = \frac{\binom{11}{5}}{\binom{10}{4}} = \boxed{\frac{11}{5}}$$