Centennial Mu Alpha Theta

April 5, 2025

Geometry Round

Do not begin until instructed to do so.

This is the Geometry Round test for the 2024 DECAGON Math Tournament. You will have 50 minutes to complete 15 problems. All problems are weighted equally, but ties will be broken based on the hardest question solved (not necessarily highest numbered question). Express all answers in simplest form. Only answers recorded on the answer sheet below will be scored. Only writing tools and plain scratch paper are allowed. Assume all questions are in base 10 unless otherwise indicated. We designed this test so that most people will not be able to finish all the questions in time, so don't worry if you are struggling! Feel free to skip questions and come back to them later.

Name: ________________________ Competitor ID: ______ Team ID: ______

1. ______________ 2. ______________ 3. ______________

4. ______________ 5. ______________ 6. ______________

7. ______________ 8. ______________ 9. ______________

10. ______________ 11. ______________ 12. ______________

13. ______________ 14. ______________ 15. ______________

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1. What is the maximum number of 2 by 1 tiles that can fit inside a 9 by 5 grid? Tiles may be rotated.

2. Point M is inside of triangle ABC. Points A', B', and C' are made by reflecting A, B, and C, respectively across point M. What is the area of the region of points M such that the area of ABC is greater than the area of $A'B'C'$?

3. A cake in the shape of a cube is painted with frosting on all but the bottom of its faces. It is then cut into 27 equal sized cubes with cuts parallel to the faces of the cube. How many of these pieces have frosting on it?

4. A square has side lengths 2025. Four congruent rectangles are placed inside the square such that two of each rectangle's sides line up with the outer square and one of each rectangle's sides line up with another rectangle. The perimeter of the inner square formed is 2020. Find the longer side length of the rectangles.

5. In triangle ORZ, it is known that $OR = 14$ and $RZ = 34$. How many integer side lengths can side OZ take?

6. A circle of radius 2 inches is internally tangent to a circle of radius 10 inches. How many revolutions will the smaller circle make if it rolls along the full circumference of the larger circle?

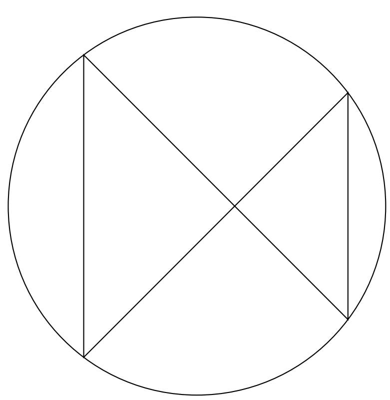
7. A cube is inscribed in a unit sphere. Jaden the fly walks the shortest distance possible along the faces of the cube from one corner to the opposite corner. He then flies back in the shortest path through the cube. Finally, he walks along the inside surface of the sphere to the opposite corner in the shortest possible distance. Find the total distance Jaden traveled.

8. In triangle ABC, $\angle A = 42°$and $\angle B = \angle C = 69°$. Points P and Q are chosen on segment AB with $P \neq B$ such that triangle APC is isosceles and CQ bisects $\angle C$. What is the measure of $\angle PCQ$?

9. When choosing a random point P on the cartesian plane, there is a $\frac{1}{2025}$ chance it is within distance d from a lattice point (a point in the cartesian plane with integer coordinates), with $d = \frac{n}{m\sqrt{\pi}}$, where n, m are relatively prime. What is $m - n$?

10. Rhombus $ABCD$ has side length 2 with $\angle A = 60°$. Altitudes are drawn from B to AD and CD, and from D to AB and BC. Find the area of the quadrilateral formed by these four altitudes.

11. Two isosceles triangles with parallel bases are inserted inside a circle of radius 10 cm as shown below. If the triangle bases are 6 cm and 8 cm away from the center of the circle, find the total area of both triangles.



12. A regular hexagon has side length 4. For each side of the hexagon, a circle inside the hexagon is tangent to it at its midpoint. All six circles have the same radius and any two adjacent circles are tangent. What is this common radius?

13. Let S_0 be the region of points bounded by a square with side length 2. For integers $n > 0$, define S_n as the set of points exactly 1 unit away from any point in S_{n-1}. Find the area bounded by S_{10}.

14. A rhombus is inscribed in a unit cube such that two of its vertices lay on opposite vertices of the cube, and its sides all lie on faces of the cube. Find the area of the rhombus.

15. In convex trapezoid $LUKA$, $LU \parallel KA$ with $LU < KA$. There exists a point P inside $LUKA$ such that $\angle LPU = 103°$, $\angle KPA = 77°$ and $\angle PLU = 73°$. Suppose KL and AU intersect at T. Find $\angle TPK$, in degrees.

Geometry Answers

1. 22
2. 0
3. 25
4. 1265
5. 27
6. 4
7. $\frac{2\sqrt{15}+6}{3} + \pi$
8. $\frac{15}{2}$
9. 44
10. $\frac{2\sqrt{3}}{3}$
11. 100
12. $\frac{2\sqrt{3}}{3}$
13. $100\pi + 84$
14. $\frac{\sqrt{6}}{2}$
15. 107

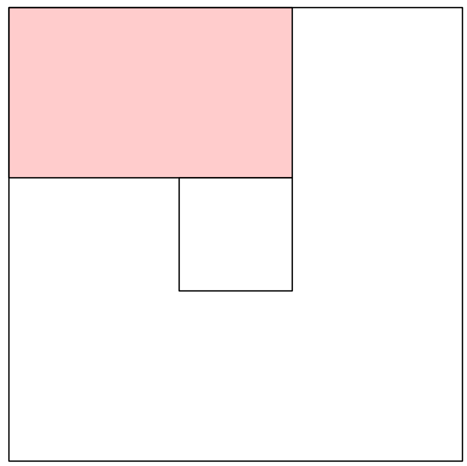
Geometry Solutions

1. The area of each 2 by 1 tile is 2 units squared, and the total area of the 9 by 5 grid is 45 units squared. Thus, grid's area can fit up to $\left\lfloor \frac{45}{2} \right\rfloor = \boxed{22}$ tiles. One way to accomplish this is by rotating the tiles such that they form an inwardly spiraling pattern, leaving only one space uncovered.

2. The answer is $\boxed{0}$. For any point M in the plane, the area of the two triangles is the same because $A'B'C'$ is simply ABC rotated $180°$ about M.

3. The answer is $\boxed{25}$. Using some imagination, the only two un-iced pieces are the middle cube and the cube below it. Thus, there are $27 - 2 = 25$ iced pieces of cake.

4. The answer is 1265.



The side length of the smaller square is $\frac{2020}{4} = 505$; then, the distance from a side of the smaller square to the larger square is $\frac{2025-505}{2} = 760$. It follows that the larger length of the rectangle is $505 + 760 = \boxed{1265}$.

5. The answer is $\boxed{27}$. By the triangle inequality, the length of OZ must be between $34 - 14 = 20$ and $34 + 14 = 48$, exclusive. There are 27 integers in this range.

6. To determine how far the smaller circle needs to roll in order to make a full revolution, we need to look at the center of the circle. Once the center has traveled a length equivalent to circumference of the smaller circle, a revolution has been completed. Since the total path that the center travels is equivalent to the circumference of a $10-2 = 8$ inch radius circle, we can find the number of revolutions by dividing this circumference by the smaller circle's circumference, giving us an answer of $\frac{16\pi}{4\pi} = \boxed{4}$ revolutions.

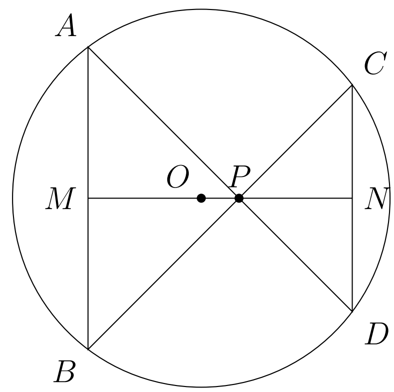
7. To find the side length s of the cube inscribed in the sphere, we can create a $45 - 45 - 90$ triangle from one corner, and with that triangles hypotenuse and an adjacent side length, we can form another right triangle. Using the Pythagorean Theorem, we can find this triangle's hypotenuse, running from two opposite corners, is equal to $\sqrt{3} \times s$, which is also equal to 2 because it is in a unit sphere. This gives $s = \frac{2}{\sqrt{3}}$. When Jaden walks the shortest distance possible long the faces of the cube, this would involve crossing two faces. If we flatten these two faces, we see this path is equivalent to the distance between two opposite corners of a rectangle with side lengths s and $2s$, which equals $\sqrt{5} \times s = \frac{2\sqrt{5}}{\sqrt{3}}$. The distance through the cube back to his original point is twice the radius $= 2$. The distance along the sphere to the opposite point is $2\pi r/2 = \pi$. Adding these together, we get $\boxed{\dfrac{2\sqrt{15}+6}{3} + \pi}$.

8. The answer is $\boxed{\dfrac{15}{2}}$. From the requirements, $\angle ACP = \angle CAP = 42°$ and $\angle ACQ = \frac{69}{2}°$. Thus, $\angle PCQ = 42° - \frac{69}{2}° = \frac{15}{2}°$.

9. To have a $\frac{1}{2025}$ chance a random point is within distance d of a lattice point, the area of a circle with radius d must be $\frac{1}{2025}$ of a unit square, both surrounding the lattice point. Therefore, $d^2\pi = \frac{1}{2025}$. Solving this equation gives us $d = \frac{1}{45\sqrt{\pi}}$, so m-n= $\boxed{44}$.

10. The answer is $\boxed{\frac{2\sqrt{3}}{3}}$. By symmetry, the quadrilateral formed by the altitudes and $ABCD$ are similar. Each of the altitudes have the same length, equal to $2\sin(60) = \sqrt{3}$. Let M be the point at which the altitude from B meets AD. Then the altitude of the smaller quadrilateral is $2 - AM = 2 - \cos(60) = 1$, meaning the ratio of the two areas is $(\frac{1}{\sqrt{3}})^2 = \frac{1}{3}$. The area of $ABCD$ is $2 * \sqrt{3}$, so the smaller quadrilateral has area $2\sqrt{3} * \frac{1}{3} = \frac{2\sqrt{3}}{3}$.

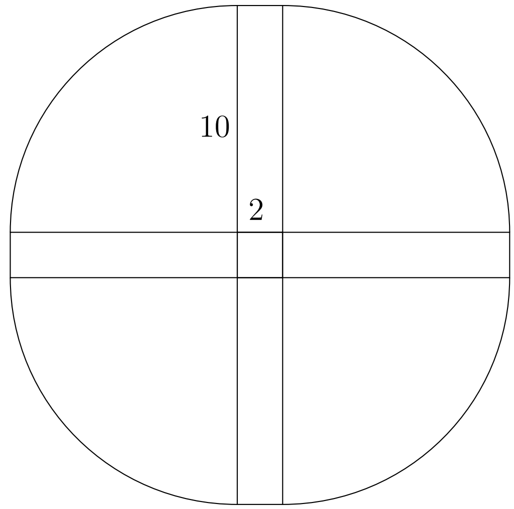
11. Since the two triangles are isosceles, we can draw a diameter through the center O that goes through the triangles' intersection point P and is perpendicular to both bases. The chord radius theorem states that this diameter bisects the bases at M and N.



We know that MO is 6 cm and ON is 8 cm. We can find the lengths of the bases of the triangles by drawing radii OB and OC. We can then use the Pythagorean theorem on the resulting right triangles to find that $MB = 8$ cm and $CN = 6$ cm, resulting in base lengths of 16 cm and 12 cm, respectively. We then observe that since AB and CD are parallel, corresponding angles cause ABP and CDP to be similar. We can thus solve for the heights of the triangles using $\frac{MP}{PN} = \frac{AB}{CD} = \frac{4}{3}$. Since $MN = MO + ON = 14$ cm, $MP = 8$ cm and $PN = 6$ cm. Finally, the total area of both triangles is $\frac{16\times8}{2} + \frac{12\times6}{2} = 64 + 36 = \boxed{100}$ cm.

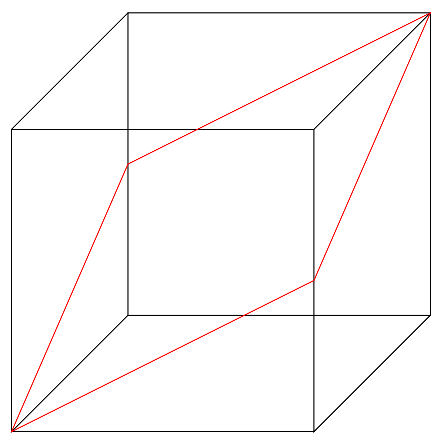
12. The answer is $\boxed{\frac{2\sqrt{3}}{3}}$. If the hexagon is split into 6 equilateral triangles, it can be seen that each circle is the incircle of their respective triangles. Since the side length of the triangles is 4, the radius is $\frac{4}{2\sqrt{3}} = \frac{2\sqrt{3}}{3}$

13. The answer is $\boxed{100\pi + 84}$. Notice that we do not need to recursively generate $S_1, S_2...$ to find the shape of S_{10}. S_{10} is simply the region of points 10 or less units away from a point in S_0.



The four arcs make a full circle of radius 10 with area 100π. Adding the four rectangles and the original square gives a total area of $100\pi + 4(2 * 10) + 2^2 = 100\pi + 84$.

14. Since two of the rhombus's vertices are on opposite vertices of the cube, the other two vertices must lay on the midpoints of opposite edges, as shown by the figure below.



The area of a rhombus is $A = \frac{d_1 d_2}{2}$ where d_1 and d_2 are the lengths of its diagonals. The length of the diagonal that connects the two opposite vertices is $\sqrt{1^2+1^2+1^2} = \sqrt{3}$ while the length of the other diagonal is simply the length of a unit square's diagonal: $\sqrt{1^2+1^2} = \sqrt{2}$. Thus the area is $A = \frac{\sqrt{3}\times\sqrt{2}}{2} = \boxed{\frac{\sqrt{6}}{2}}$.

15. The answer is 107. There exists a dilation at T that sends segment LU to AK, and suppose that this dilation also sends P to a point Q. Then since two triangles that can be dilated are similar, $\triangle LPU \sim \triangle AQK$. Thus, $\angle AQK = \angle LPU = 103°$. Now, note

$$\angle APK + \angle AQK = 77° + 103° = 180°$$

so it follows that $APKQ$ is a cyclic quadrilateral, as the opposite angles add to 180°. Now, since Q is the result of a dilation centered at T on P, it follows that Q lies on TP. Hence

$$\angle TPA = 180 - \angle QPA$$

it suffices to find $\angle QPA$, but it subtends arc QA in the circle passing through A, P, K and Q. Thus,

$$\angle QPK = \angle QAK = \angle PLU = 73°$$

where $\angle QAK = \angle PLU$ from recalling $\triangle LPU \sim \triangle AQK$. Thus, the answer is $180° - 73° = \boxed{107}°$.