Centennial Mu Alpha Theta

April 5, 2025

Team Round

Do not begin until instructed to do so.

This is the Team Round test for the 2024 DECAGON Math Tournament. You will have 30 minutes to complete 10 problems. All problems are weighted equally, but ties will be broken based on the hardest question solved (not necessarily highest numbered question). Express all answers in simplest form. Only answers recorded on the answer sheet below will be scored. Only writing tools and plain scratch paper are allowed. Assume all questions are in base 10 unless otherwise indicated. We designed this test so that most people will not be able to finish all the questions in time, so don't worry if you are struggling! Feel free to skip questions and come back to them later.

Name: ________________________ Competitor ID: ______ Team ID: ______

1. ______________ 2. ______________ 3. ______________

4. ______________ 5. ______________ 6. ______________

7. ______________ 8. ______________ 9. ______________

10. ______________

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1. Five students in a math class- A, B, C, D, E -are to be seated around a circular table. The following pairs cannot sit next to each other:

   - A and B
   - B and C
   - C and D
   - D and E
   - E and A

   Determine the number of distinct seating arrangements possible under these constraints. Rotations and reflections are considered distinct.

2. In Antarctica, there are four types of coins worth 4, 7, 9, and 11. Suppose that the maximum coin value that cannot be obtained by using one or more coins is M. What is M?

3. Alice and Bob are removing stones from a pile. In a turn, they may remove one, two, three, or four stones, but cannot remove more stones than the amount present. The first player who cannot make a move loses. Alice moves first.

   Let S be the set of positive integers n such that when starting with n stones, Bob can guarantee a win regardless of Alice's moves. Find the 100th smallest positive integer in S.

4. Given that $x^4 - ax^3 + bx^2 + cx + 2025 = 0$ only has positive solutions, find the smallest possible value of a.

5. The number 11111111... (50 1s) in base n is divisible by 999999.... (25 9s) also in base n. What is the remainder when n^{2025} is divided by 9?

6. Triangle ABC has $AB = 3$, $BC = 7$, and $CA = 5$. The angle bisector of BAC meets the circumcircle of ABC at $P \neq A$. What is the length of PA?

7. In triangle $\triangle ABC$, we have $AB = AC = 5$ and $BC = 6$. There is a point P on minor arc BC of the circumcircle of $\triangle ABC$ such that $AP = 6$. Let AP intersect BC again at Q. Find AQ.

8. Consider the polynomial $P(x) = x^2 - 2$. It is given that a root of $P(P(x)) - 1$ is $2\cos(k^\circ)$, for some positive integer $k < 45^\circ$. Find k.

9. Quadrilateral $ABCD$ is inscribed in a circle with radius 13. Lines AB and CD are parallel, lines BC and AD are perpendicular, and $AB = 7\sqrt{2}$. What is the area of $ABCD$?

10. A game board is set up such that Abby's character and Bob's character start at square 0 and to win is to get to square 3. The subsequent squares can be reached by moving forward. Every round, an unfair coin is flipped, with a 0.7 probability of landing on heads and a 0.3 probability of landing on tails. If the coin lands on heads, Abby's character moves forward one square and if the coin lands on tails, Bob's character moves forward one square. Because of Abby's advantage, Bob makes a rule so that if Abby and Bob end up on the same positive numbered square, Abby's character must go back one square. What is the probability that Abby wins?

Team Answers

1. 10
2. 10
3. 500
4. $12\sqrt{5}$
5. 8
6. 8
7. $\frac{25}{6}$
8. 15
9. 120
10. $\frac{4459}{10000}$

Team Solutions

1. First there are 5 ways to choose where A sits. Then B must sit in one of the two spots not adjacent to A; there are 2 ways to choose this.

   Since C is not adjacent to B then it turns out C only has one place to sit; similarly, D and E are forced. Hence the answer is $5 \times 2 = \boxed{10}$.

2. We notice that 10 is not achievable. However, $11, 12, 13, 14$ are achievable, and by using coins with value 4 any other value is achievable. Thus $M = \boxed{10}$.

3. We claim that starting with n stones, Bob wins if and only if n is divisible by 5.

   When n is divisible by 5, then whenever Alice removes x stones, Bob should just remove $5 - x$ stones. Thus after Bob makes a move, the number of stones is still divisible by 5. So if there are zero stones left, Bob must have made the last move, so Alice loses.

   When n is not divisible by 5, Suppose that n leaves a remainder of a when divided by 5. Alice can then take those a stones, so there are now $n - a$ stones. Alice now repeats Bob's strategy described above, and by similar reasoning Alice can win.

   Thus the answer is just the 100th positive integer divisible by 5. The answer is $\boxed{500}$.

4. Since we are given a quartic equation, and the solutions are all positive, we can represent the solutions as positive reals p, q, r, s. By Vieta's Theorem, $p + q + r + s = -(-a) = a$ and $pqrs = 2025$. Based on AM-GM, $p + q + r + s \geq 4\sqrt[4]{pqrs}$. Therefore the minimum value of $p + q + r + s$ is $\boxed{12\sqrt{5}}$.

5. 1111... (50 1s) in base n is equal to

$$n^{49} + n^{48} + \cdots + n + 1 = \frac{n^{50} - 1}{n - 1} = \frac{(n^{25} - 1)(n^{25} + 1)}{n - 1}$$

   and 9999... (25 9s) is equal to

$$9(n^{24} + n^{23} ... + n + 1) = \frac{9(n^{25} - 1)}{n - 1}$$

   Note $\frac{n^{50}-1}{n-1}$ is divisible by $9 \cdot \frac{n^{25}-1}{n-1}$ so $n^{25} + 1$ divisible by 9. Therefore, $n^{25} \equiv -1 \pmod 9$ so $n^{2025} \equiv -1 \pmod 9 \equiv \boxed{8} \pmod 9$.

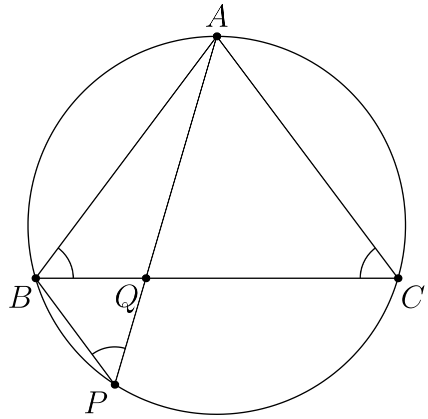
6. The answer is $\boxed{8}$. Using Law of Cosines, note

$$\cos \angle A = \frac{3^2 + 5^2 - 7^2}{2 \cdot 3 \cdot 5} = \frac{-15}{30} = -\frac{1}{2}$$

   so it follows that $\angle A = 120°$. Since opposite angles in a cyclic quadrilateral add up to $180°$, $\angle BPC = 60°$. From angle chasing, we have that $\angle BAP = \angle BCP = 60°$. Therefore, $\triangle BPC$ is equilateral. If we rotate $\triangle APC$ such that PC is on PB, notice that we get an equilateral triangle. Therefore, $PA = PB + PC = 3 + 5 = 8$.

7. Chasing angles,

$$\angle APB = \angle ACB = \angle ABC = \angle ABQ$$



and we also have $\angle BAP = \angle QAB$. Thus, $\triangle APB \sim \triangle ABQ$, and so

$$\frac{AP}{AB} = \frac{AB}{AQ} \implies AQ = \frac{AB^2}{AP} = \frac{25}{6}$$

thus the answer is $\boxed{\frac{25}{6}}$.

8. The answer is 15. Consider some complex number t, with $|t| = 1$, and note that then

$$\bar{t} = \frac{|t|^2}{t} = \frac{1}{t}$$

it follows that $t + \frac{1}{t} = t + \bar{t} = 2 \cdot \text{Re}(t)$. Thus if $t = e^{i\theta}$ which is possible due to Euler, then $t + \frac{1}{t} = 2\cos\theta$. Plug $x = t + \frac{1}{t}$, so then

$$P\left(t + \frac{1}{t}\right) = \left(t + \frac{1}{t}\right)^2 - 2 = t^2 + \frac{1}{t^2}$$

It follows by similar logic that

$$P\left(P\left(t + \frac{1}{t}\right)\right) = P\left(t^2 + \frac{1}{t^2}\right) = t^4 + \frac{1}{t^4}$$

Thus, we have that $t + \frac{1}{t}$ is a root only if $t^4 + \frac{1}{t^4} = 1$. Note that $|t^4| = |t|^4 = 1$, so by our previous logic then

$$t^4 + \frac{1}{t^4} = 2 \cdot \text{Re}(t^4)$$

and since $t = e^{i\theta}$ then $t^4 = e^{i \cdot 4\theta}$, and so $\text{Re}(t^4) = \cos(4\theta)$. Thus, we must have

$$1 = t^4 + \frac{1}{t^4} = 2\cos(4\theta) \implies \cos(4\theta) = \frac{1}{2}$$

Recall that we are looking for a solution with $k < 45$, i.e. $\theta < 45°$. Then the only possible θ is $15°$, as claimed.

9. Since $ABCD$ is a trapezoid inscribed in a circle, it is isosceles with $BC = AD$. Furthermore, $AB = 7\sqrt{2}$ and arc $AB < 90$, so $AB < CD$. Let lines BC and AD intersect at point X. AXB and CXD are isosceles right triangles because $\angle AXB = 90$ and trapezoid $ABCD$ is isosceles. In particular, the legs of AXB have length 7 from the hypotenuse AB. Let M and N be the midpoints of BC and AD respectively, and the center of the circle be point O. Also, let $x = BM = AN$ so that $ONXM$ is a square with side length $x + 7$. Then, OMB is a right triangle with legs to x and $x + 7$ and hypotenuse 13. Pythagorean theorem gives $(x - 5)(x + 12)$ and the positive root gives $x = 5$, so triangle CXD has legs of length $7 + 2x = 17$. The area of $ABCD$ can be found by subtracting the areas of CXD and AXB, giving an area of $\frac{1}{2}(17^2 - 7^2) = \boxed{120}$.

10. We are going to craft a state diagram. Assume (a, b) represents the probability that we achieve that state, with a representing the position of Abby's character and b representing the position of Bob's character. Note that we must never let $b \geq a$, otherwise Abby will lose. From (a, b), we go to $(a+1, b)$ with probability 0.7 and $(a, b+1)$ with probability 0.3. From $(0, 0)$ being equal to 1, we can derive that $(3, 0)$ and $(3, 1)$ have probability $\frac{343}{1000}$ and $\frac{1029}{10000}$ respectively. Adding them up gives us $\boxed{\frac{4459}{10000}}$.