Centennial Mu Alpha Theta

April 11, 2026

Algebra and Number Theory Round

Do not begin until instructed to do so.

This is the Algebra and Number Theory Round test for the 2026 DECAGON Math Tournament. You will have 45 minutes to complete 10 problems. All problems are weighted equally, but ties will be broken based on the hardest question solved (not necessarily highest numbered question). Express all answers in simplest form. Only answers recorded on the answer sheet below will be scored. Only writing tools and plain scratch paper are allowed. Assume all questions are in base 10 unless otherwise indicated. We designed this test so that most people will not be able to finish all the questions in time, so don't worry if you are struggling! Feel free to skip questions and come back to them later.

Name: ________________________ Competitor ID: ______ Team ID: ______

1. ______________ 2. ______________ 3. ______________

4. ______________ 5. ______________ 6. ______________

7. ______________ 8. ______________ 9. ______________

10. ______________

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1. A number n is chosen between 1 and 100. Exactly two of the following are true:

   - The number is a perfect square
   - The number is prime
   - The number is divisible by 17

   What is n?

2. How many perfect squares from the set of numbers (0,100] can be written as the product of 2 distinct positive integers, where neither of the integers are 1?

3. What is the sum of the x-coordinates of the intersection points of the equations $y^2 = 4x + 140$ and $y = x$?

4. Let f be a function from positive integers to positive integers such that $f(n) = 3n + 1$ when n is odd and $f(n) = n/2$ when n is even. If $f(f(n)) = 6$, then find n.

5. Kevin and Devin both shovel snow. Kevin started out with 500 dollars and makes 30 dollars for each driveway he shovels, but he spends 50 dollars every 10th driveway he shovels to buy a new shovel. Devin started out with 300 dollars and makes 40 dollars for each driveway he shovels, but he spends 20 dollars every 9th driveway he shovels. Let n be a positive number. After Kevin and Devin shovel n driveways, Devin now has the same amount of money as Kevin. What is the least possible value of n?

6. Let a and b be positive integers. Find the sum of all possible values of a if $\frac{1}{ab} + \frac{1}{a} + \frac{1}{b} = \frac{1}{2}$.

7. Find the product of all positive integers n such that $\frac{n(n+1)}{2} - 1$ is prime.

8. What is the greatest positive integer n such that if I choose any 5 positive integers less than n, I can guarantee there are two numbers I chose x and y such that $x < y$ and $3x \geq y$?

9. Let n be a positive integer. Find the largest k such that we can guarantee that k divides $n(n+1)(n+4)(n+7)(n+8)$.

10. Let $S = \{2^a 3^b \mid a, b \geq 0\}$. Compute

$$\sum_{n \in S} \frac{n-1}{n^2}$$

Algebra and Number Theory Answers

1. 17
2. 5
3. 4
4. 24
5. 17
6. 20
7. 6
8. 121
9. 120
10. 3/2

Algebra and Number Theory Solutions

1. The first two statements cannot be both true. If the first and the third are true then n is at least 17^2, which is greater than 100. So the second and third are true. This happens when $n = 17$.

2. The perfect squares in the set (0, 100] are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100. Those that can be written as the product of 2 distinct integers that are not equal to 1 will have non-prime roots. As 1, 4, 9, 25, and 49 have prime roots, they are eliminated. 16, 36, 64, 81, 100 are all viable, so the answer is 5.

3. By substituting y for x (from the equation $y = x$), we get $x^2 = 4x+140$, which can then be rearranged to $0 = x^2 - 4x - 140$, or $0 = (x - 14)(x + 10)$, which can be solved for $x = -10, 14$ and the sum is $14 - 10 = 4$.

4. First if $f(k) = 6$ then $k = 12$, since $3k + 1$ cannot equal 6 for an integer k. So $f(n) = 12$, and by similar logic $n = \boxed{24}$.

5. Let number of driveways shoveled for both be n, and let $[x]$ denote the floor of x. Kevin makes $500 + 30n - 50 * [n/10]$ and Devin makes $300 + 40n - 20 * [n/9]$ dollars. If $n < 9$ then the floors are both zero; but no solutions are valid when we try to solve $500 + 30n = 300 + 40n$, as we get $n = 20$, contradiction. Note $n = 10$ doesn't work, so assume $11 \le n < 18$. Kevin makes $450 + 30n$ and Devin makes $280 + 40n$, and solving gives $n = 17$ which works.

6. If we multiply the whole expression by $2ab$, we get $2 + 2b + 2a = ab \to ab - 2a - 2b - 2 = 0$. Using Simon's favorite factoring theorem, we get $(a - 2)(b - 2) = 6$. By looking at the factors of 6, it is clear that a can be equal to 3, 4, 6, or 8. Thus, the sum is $\boxed{20}$.

7. Turning this into one fraction, we get $\frac{n^2+n-2}{2} = \frac{(n+2)(n-1)}{2}$. Notice that $n = 2, 3$ gives us prime numbers. For $n \ge 4$, both $\frac{n+2}{2}$ and $\frac{n-1}{2}$ are greater than 1, thus the number would be representable as a product of two integers which are not equal to 1. Thus, it would be composite. Therefore, the only possible n are 2 and 3, and, thus, the product is $\boxed{6}$.

8. To choose the five smallest numbers $a < b < c < d < e$ that break this, then $3a < b$ and $3b < c$ and et cetera. But $a \ge 1$ means $b \ge 4$ and $c \ge 13$ and $d \ge 40$ and $e \ge 121$. So if $n = \boxed{121}$, then it isn't possible to find such a set.

9. Let $f(n) = n(n + 1)(n + 4)(n + 7)(n + 8)$. We only need to see how many of each prime divides $f(n)$ across all n. It is not guaranteed that primes above 5 divide the product. Simply looking at the factorization, we don't have enough different numbers to cover all possible modulo residues of primes greater than 5. Additionally, $f(1) = 720$, which does not have any primes besides 2, 3, and 5.

   Let us first look at 5. Taking a mod 5, we see that $f(n) = n(n+1)(n+2)(n+3)(n+4)$. This means that we are always guaranteed a 5. Since $f(1)$ only contains one 5, we know there cannot be more guaranteed 5s in the prime factorization of k.

   Looking at 3, taking a mod 3, we see that $f(n) = n(n + 1)(n + 1)(n + 1)(n + 2)$. We are, thus, guaranteed at least one 3. If $n = 3$, $f(n)$ would only have one 3, thus we know there cannot be more guaranteed 3s in the prime factorization of k.

   If n is even, we have 3 guaranteed 2s, and $n = 2$ can give this minimum. If n is odd, we also have 3 guaranteed 2s. This is because either $n + 1$ or $n + 7$ must also be divisible by 4, as an odd number must be in the form $4a+1$ or $4a+3$. It turns out $n = 3$ gives us two 3s. Thus, $k = 2^3 \times 3 \times 5 = \boxed{120}$.

10. We can rewrite the expression to be $\sum_{n \in S}[\frac{1}{n} - \frac{1}{n^2}]$. Let $A_k = \{2^{ka} \mid a \ge 0\}$ and $B_k = \{2^{kb} \mid b \ge 0\}$. Notice that $\sum_{n \in S} \frac{1}{n} = \left(\sum_{n \in A_1} \frac{1}{n}\right) \times \left(\sum_{n \in B_1} \frac{1}{n}\right)$. Using the summation for an infinite geometric sequence, we have that $\sum_{n \in S} \frac{1}{n} = \frac{1}{1-\frac{1}{2}} \times \frac{1}{1-\frac{1}{3}} = 3$. Notice that, by the same logic, $\sum_{n \in S} \frac{1}{n} = \left(\sum_{n \in A_2} \frac{1}{n}\right) \times \left(\sum_{n \in B_2} \frac{1}{n}\right) = \frac{1}{1-\frac{1}{4}} \times \frac{1}{1-\frac{1}{9}} = \frac{3}{2}$. Thus, $\sum_{n \in S}[\frac{1}{n} - \frac{1}{n^2}] = \boxed{\frac{3}{2}}$